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Interpreting Multilevel Models for Longitudinal Data

Outline

- Linear Trajectory
- Centering
- Nonlinear Trajectory
- Models with Covariates

Linear trajectory

- Use time-related variables as L1 predictors
  - Age, Grade, Trials
- The interpretation from previous lectures is still applied here.

Linear Trajectory

$$Y_i = \beta_0 + \beta_1 t_i + e_i$$

- Model for a single participant measured at different time points
- Y_i is the DV value at measurement i
- t_i is the value of time at measurement i
- β_0 is the DV value when the time variable is 0
- β_1 is the change in DV value when time increases by 1

Linear Trajectory

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j} \qquad \beta_{1j} = \gamma_{10} + u_{1j}$$

- Measurements (i) are nested in Participants (j)
- Y_{ij} is the DV value at Measurement *i* in Participant *j*
- t_{ij} is the time that Measurement *i* in Participant *j* was obtained

Linear Trajectory

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j} \qquad \beta_{1j} = \gamma_{10} + u_{1j}$$

- β_{0j} is the DV value of Participant *j* when time is 0.
  - If time = 0 is the starting time point, β_{0j} can be interpreted as the initial status of Participant *j*

Linear Trajectory

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j} \qquad \beta_{1j} = \gamma_{10} + u_{1j}$$

- β_{1j} is the increase in DV when time increases by 1 in Participant *j*, or the rate of change in Participant *j*.

Linear Trajectory

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j} \qquad \beta_{1j} = \gamma_{10} + u_{1j}$$

- γ_{00} is the average DV values across participants when time = 0.
- γ_{10} is the average rate of change across participants.

Linear Trajectory

L1: $Y_{ij} = \beta_{0j} + \beta_{1j} t_{ij} + e_{ij} \quad e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$

$\beta_{1j} = \gamma_{10} + u_{1j}$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix} \right)$$

- τ_{00} is the variance of DV value at time = 0 across participants
  - If time = 0 is the starting time point, τ_{00} is the variance of initial status across participants

Linear Trajectory

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$ $\quad e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$

$\beta_{1j} = \gamma_{10} + u_{1j}$ $\quad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$

- τ_{11} is the variance of the rate of change across participants
- τ_{10} is the covariance between the intercepts (or initial status if time = 0 is starting point) and rate of change.

Linear Trajectory

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j} t_{ij} + e_{ij}$ $\quad e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$

$\beta_{1j} = \gamma_{10} + u_{1j}$ $\quad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$

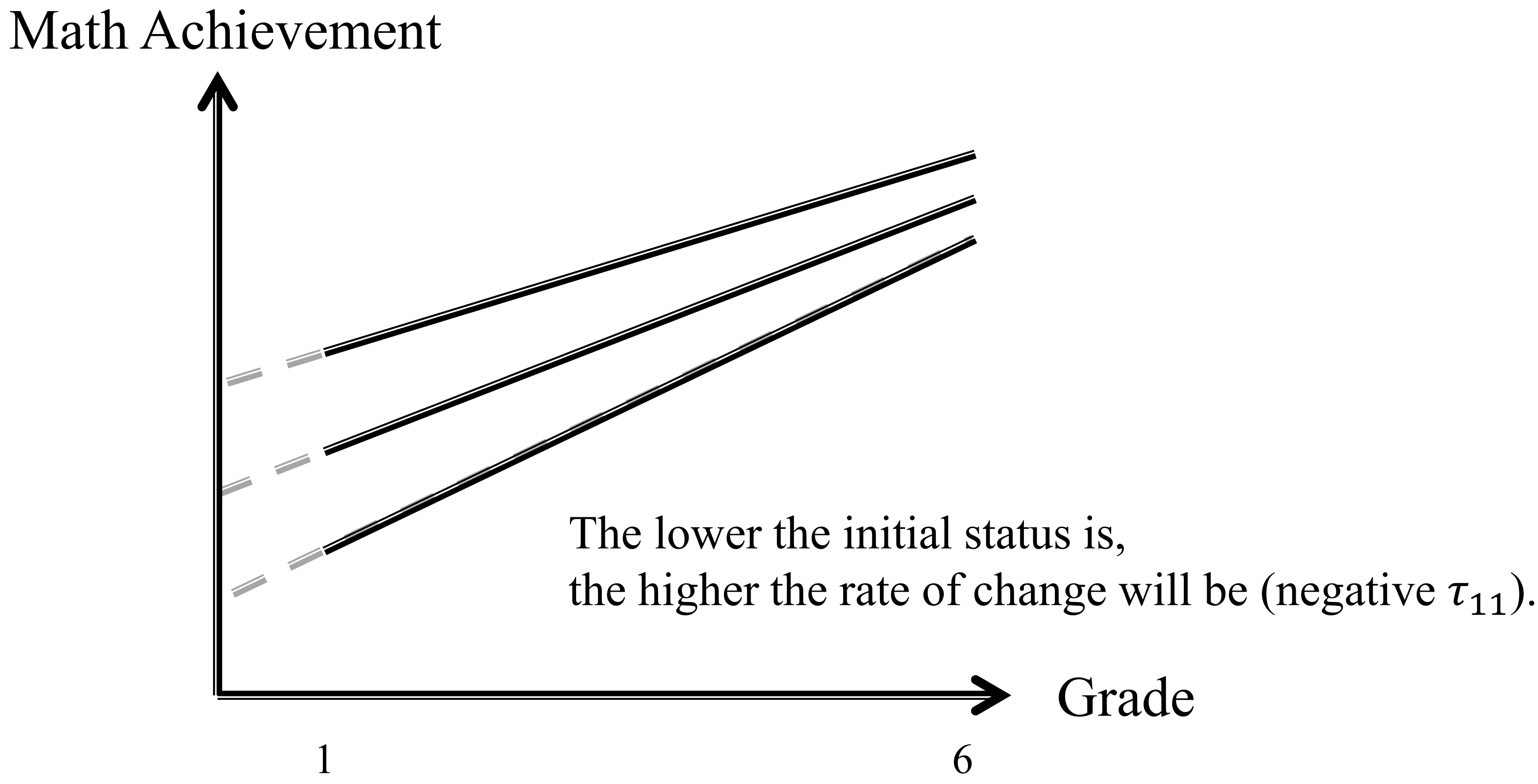
- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6)

L1: $Y_{ij} = \beta_{0j} + \beta_{1j} t_{ij} + e_{ij}$ $\quad e_{ij} \sim N(0, 50)$

L2: $\beta_{0j} = 50 + u_{0j}$

$\beta_{1j} = 2 + u_{1j}$ $\quad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 25 & \\ -0.25 & 0.25 \end{bmatrix}\right)$

Linear Trajectory



Linear Trajectory

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$ $\qquad e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$

$\beta_{1j} = \gamma_{10} + u_{1j}$ $\qquad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$

- Y_{ij} = Positive affect at Measurement *i* for Participant *j*; t_{ij} = Hour of a day (8:00-23:00)

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + e_{ij}$ $\qquad e_{ij} \sim N(0, 50)$

L2: $\beta_{0j} = 50 + u_{0j}$

$\beta_{1j} = -1 + u_{1j}$ $\qquad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 25 & \\ 0 & 2 \end{bmatrix}\right)$

Centering

- In previous example, the intercepts could be not meaningful.
- If the value of 0 is not observed, the interpretation of intercepts could be inaccurate.
  - Predict the value at 0:00 when positive affect is measured between 8:00 to 23:00.

Centering

- Time can be centered such that intercepts are meaningful
  - Starting point
  - End point
- Group- or Grand-mean centering is not recommended for time-related variables.

Centering

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + e_{ij}$ $e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$ $\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$

$\beta_{1j} = \gamma_{10} + u_{1j}$

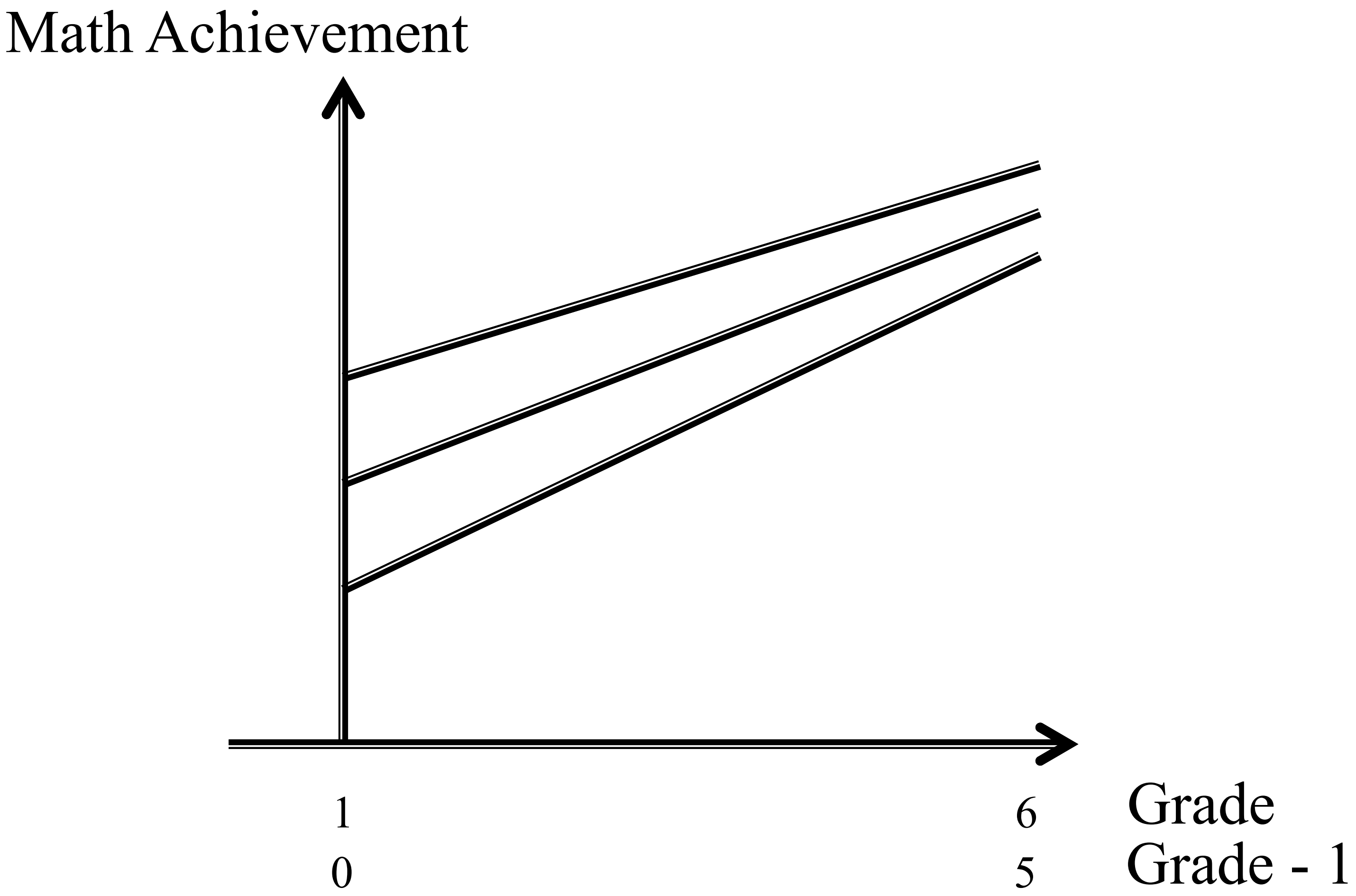
- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6)

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + e_{ij}$ $e_{ij} \sim N(0, 50)$

L2: $\beta_{0j} = 52 + u_{0j}$ $\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 & \\ -0.20 & 0.25 \end{bmatrix}\right)$

$\beta_{1j} = 2 + u_{1j}$

Centering



Centering

- What do all parameters represent?

$$\text{L1:} \quad Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 12) + e_{ij} \qquad e_{ij} \sim N(0, \sigma_e^2)$$

$$\text{L2:} \quad \beta_{0j} = \gamma_{00} + u_{0j} \qquad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

- Y_{ij} = Positive affect at Measurement *i* for Participant *j*; t_{ij} = Hour of a day (8:00-23:00)

$$\text{L1:} \quad Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 12) + e_{ij} \qquad e_{ij} \sim N(0, 50)$$

$$\text{L2:} \quad \beta_{0j} = 40 + u_{0j} \qquad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 36 & \\ 0 & 2 \end{bmatrix}\right)$$

$$\beta_{1j} = -1 + u_{1j}$$

Centering

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 2000) + e_{ij}$ $e_{ij} \sim N(0, \sigma_e^2)$

L2: $\beta_{0j} = \gamma_{00} + u_{0j}$ $\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{bmatrix}\right)$

$\beta_{1j} = \gamma_{10} + u_{1j}$

- Y_{ij} = National GDP at Measurement *i* for Country *j* (in $Billion); t_{ij} = Year (2000-2012)

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 2000) + e_{ij}$ $e_{ij} \sim N(0, 50)$

L2: $\beta_{0j} = 250 + u_{0j}$ $\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1000 & \\ 80 & 100 \end{bmatrix}\right)$

$\beta_{1j} = 30 + u_{1j}$

Nonlinear trajectory

- Any transformation on the time-related variables can be used
  - Quadratic

$$Y_{ij} = \beta_{0j} + \beta_{1j} t_{ij} + \beta_{2j} t_{ij}^2 + e_{ij}$$

  - Cubic

$$Y_{ij} = \beta_{0j} + \beta_{1j} t_{ij} + \beta_{1j} t_{ij}^2 + \beta_{3j} t_{ij}^3 + e_{ij}$$

Nonlinear trajectory

- Any transformation on the time-related variables can be used
  - Exponential

$$Y_{ij} = \beta_{0j} \exp\left(\beta_{1j} \cdot t_{ij}\right) + e_{ij}$$

  - Sinusoids

$$Y_{ij} = \beta_{0j} + \beta_{1j} \sin\left(\beta_{2j} \cdot t_{ij}\right) + e_{ij}$$

Nonlinear trajectory

- Any transformation on the time-related variables can be used
  - Spline (Piecewise Linear Model)

$$Y_{ij} = \beta_{0j} + \beta_{1j}t_{1ij} + \beta_{2j}t_{2ij} + e_{ij}$$

where $t_{ij} = t_{1ij} + t_{2ij}$

t_{ij}	0	1	2	3	4	5	6	7	8
t_{1ij}	0	1	2	3	4	4	4	4	4
t_{2ij}	0	0	0	0	0	1	2	3	4

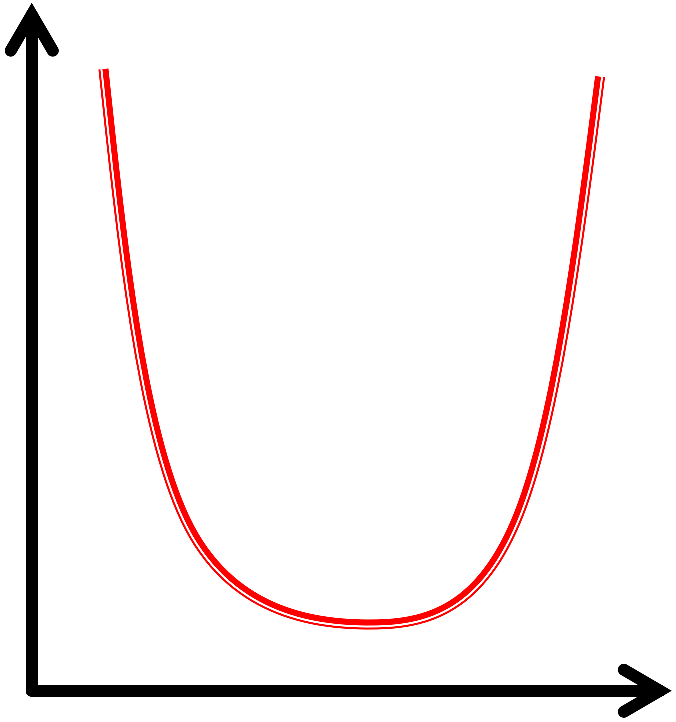
Nonlinear trajectory

$$Y_{ij} = \beta_{0j} + \beta_{1j}t_{ij} + \beta_{2j}t_{ij}^2 + e_{ij}$$

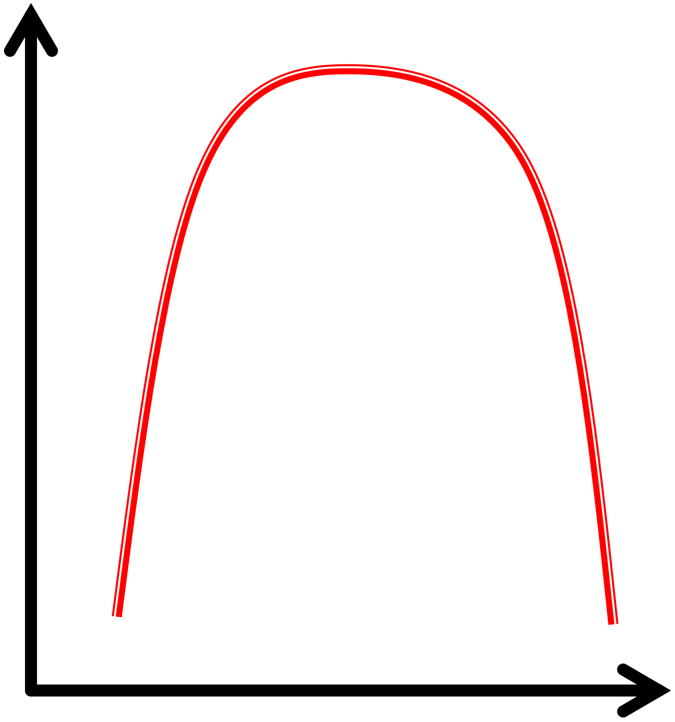
- Quadratic
- β_{2j} is the change in the rate of change when t increases by 1
  - $\beta_{2j} > 0$: Concave up
  - $\beta_{2j} < 0$: Concave down

Nonlinear trajectory

Concave Up

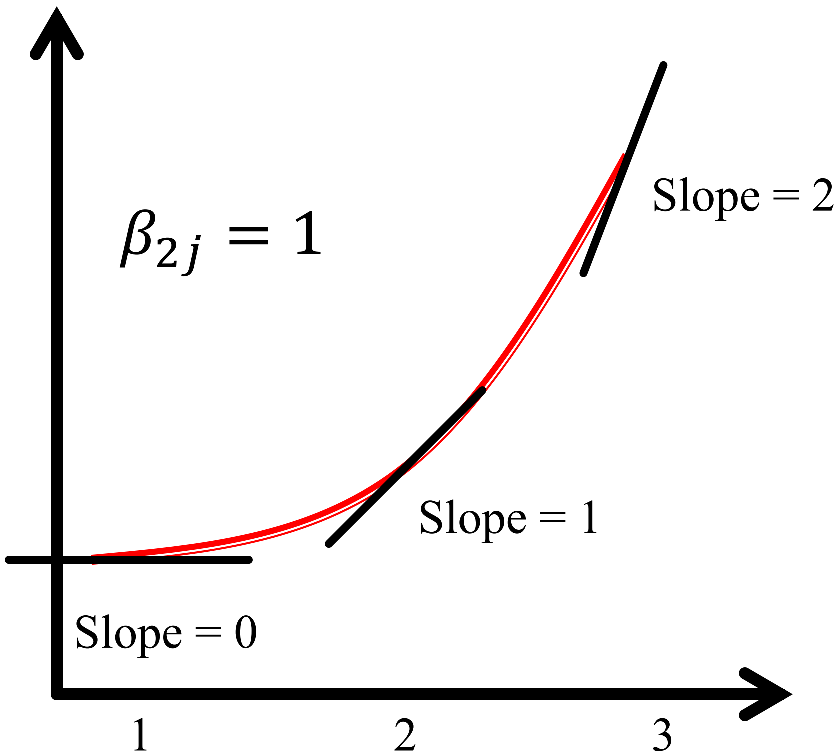


Concave Down

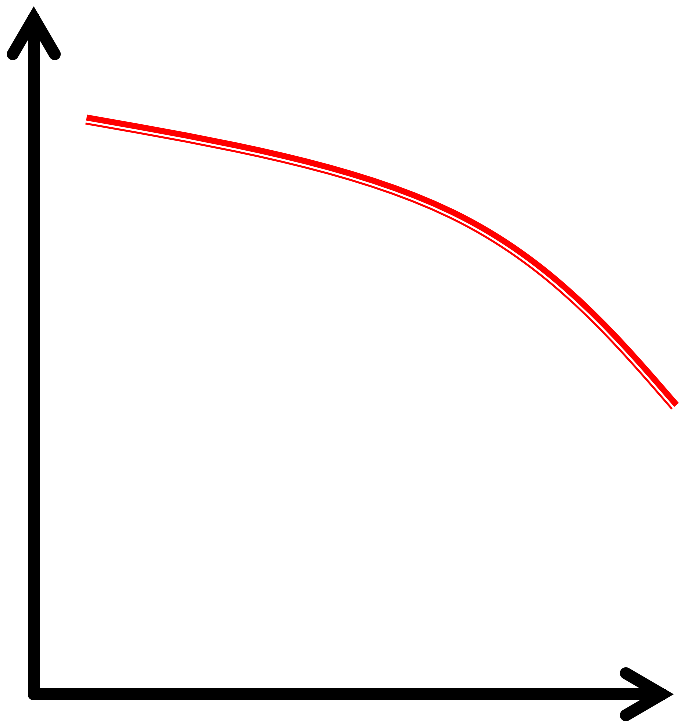


Nonlinear trajectory

Concave Up



Concave Down



Nonlinear trajectory

$$Y_{ij} = \beta_{0j} \exp\left(\beta_{1j} \cdot t_{ij}\right) + e_{ij}$$

- Exponential
- β_{0j} is the expected value of DV when $t = 0$ in Participant j.
- β_{1j} is the rate of decreasing toward or increasing away from 0.

Nonlinear trajectory

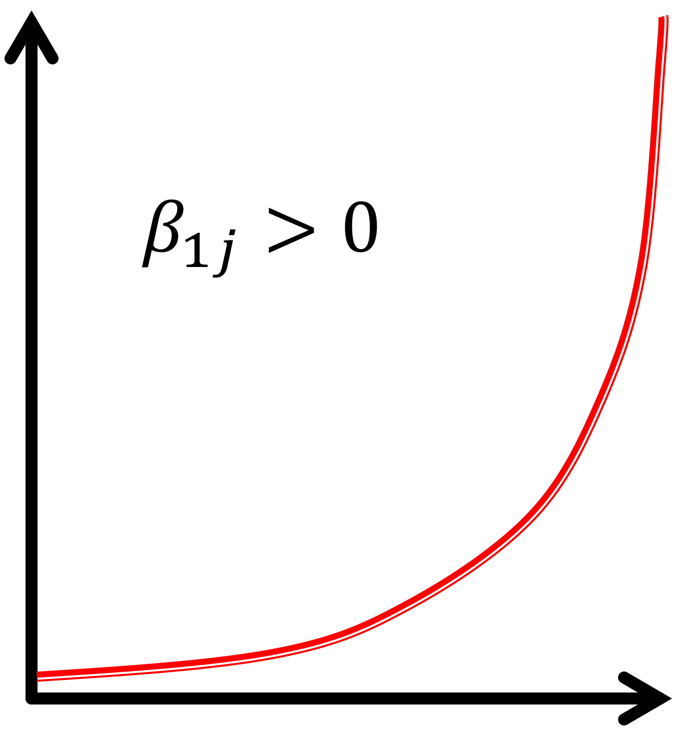
$$Y_{ij} = \beta_{0j} \exp(\beta_{1j} \cdot t_{ij}) + e_{ij}$$

- Exponential
- The proportion of predicted Y at $t = c + 1$ and Y at $t = c$ is $\exp(\beta_{1j})$

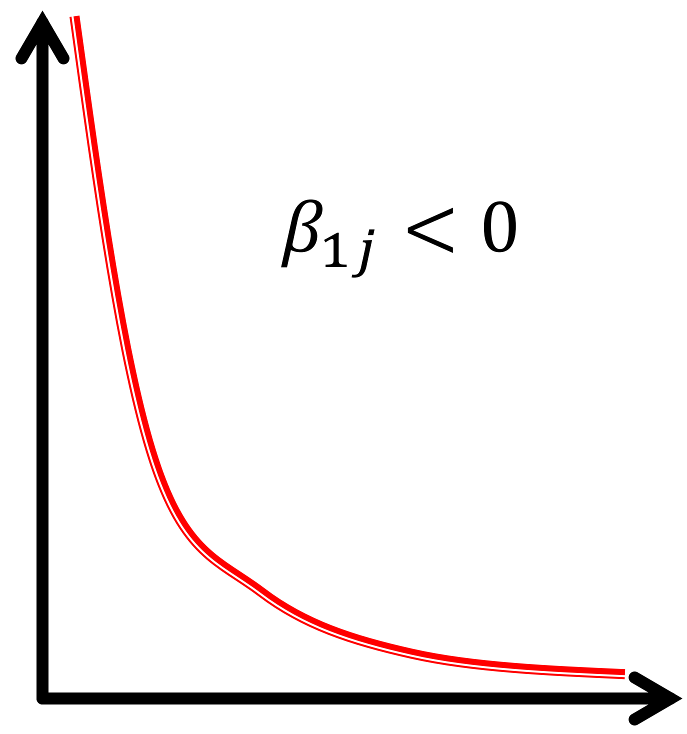
$$\frac{E(Y_{ij} | t_{ij} = c + 1)}{E(Y_{ij} | t_{ij} = c)} = \exp(\beta_{1j})$$

Nonlinear trajectory

Exponential Growth



Exponential Decay



Nonlinear trajectory

$$Y_{ij} = \beta_{0j} \exp\left(\beta_{1j} \cdot t_{ij}\right) + e_{ij}$$

- This model cannot be estimated by usual MLM program because of the nonlinearity in parameters.
  - See PROC NLMIXED in SAS

Nonlinear trajectory

$$Y_{ij} = \beta_{0j} + \beta_{1j} \sin\left(\beta_{2j} \cdot \left(t_{ij} - \beta_{3j}\right)\right) + e_{ij}$$

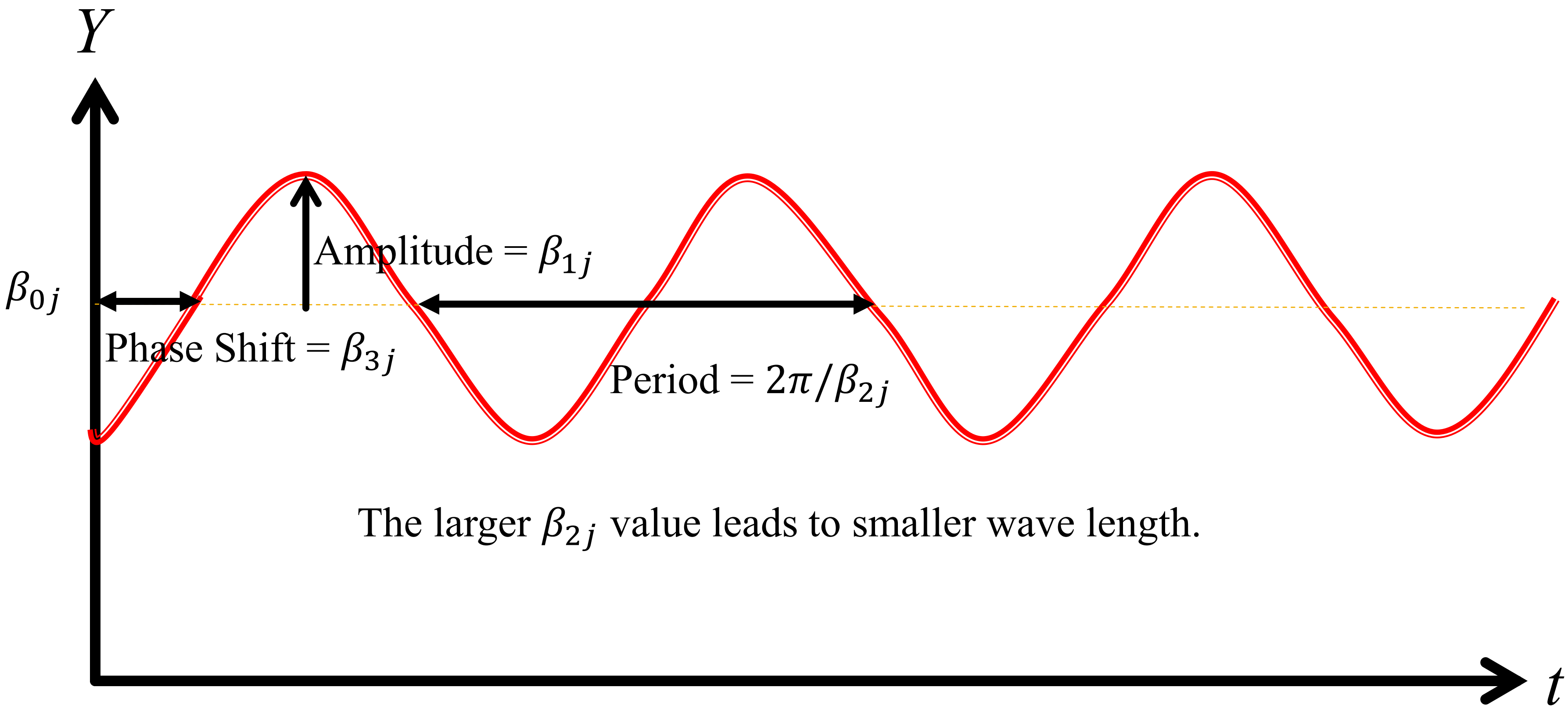
- Sinusoids
- β_{0j} is the baseline value of DV in Participant *j*.
- β_{1j} is the amplitude of the wave in Participant *j*.
- β_{2j} is a function of frequency of the wave in Participant *j*.

Nonlinear trajectory

$$Y_{ij} = \beta_{0j} + \beta_{1j} \sin\left(\beta_{2j} \cdot \left(t_{ij} - \beta_{3j}\right)\right) + e_{ij}$$

- Sinusoids
- β_{3j} is the phase shift for Participant j.

Nonlinear trajectory



Nonlinear trajectory

$$Y_{ij} = \beta_{0j} + \beta_{1j} \sin\left(\beta_{2j} \cdot \left(t_{ij} - \beta_{3j}\right)\right) + e_{ij}$$

- This model can combine with other trajectory such as linear trend
- The model cannot be estimated by usual MLM.
  - See PROC NLMIXED in SAS

Nonlinear Trajectory

- What do all parameters represent?

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}(t_{ij} - 1)^2 + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$
$$\beta_{2j} = \gamma_{20} + u_{2j}$$

- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6)

Nonlinear Trajectory

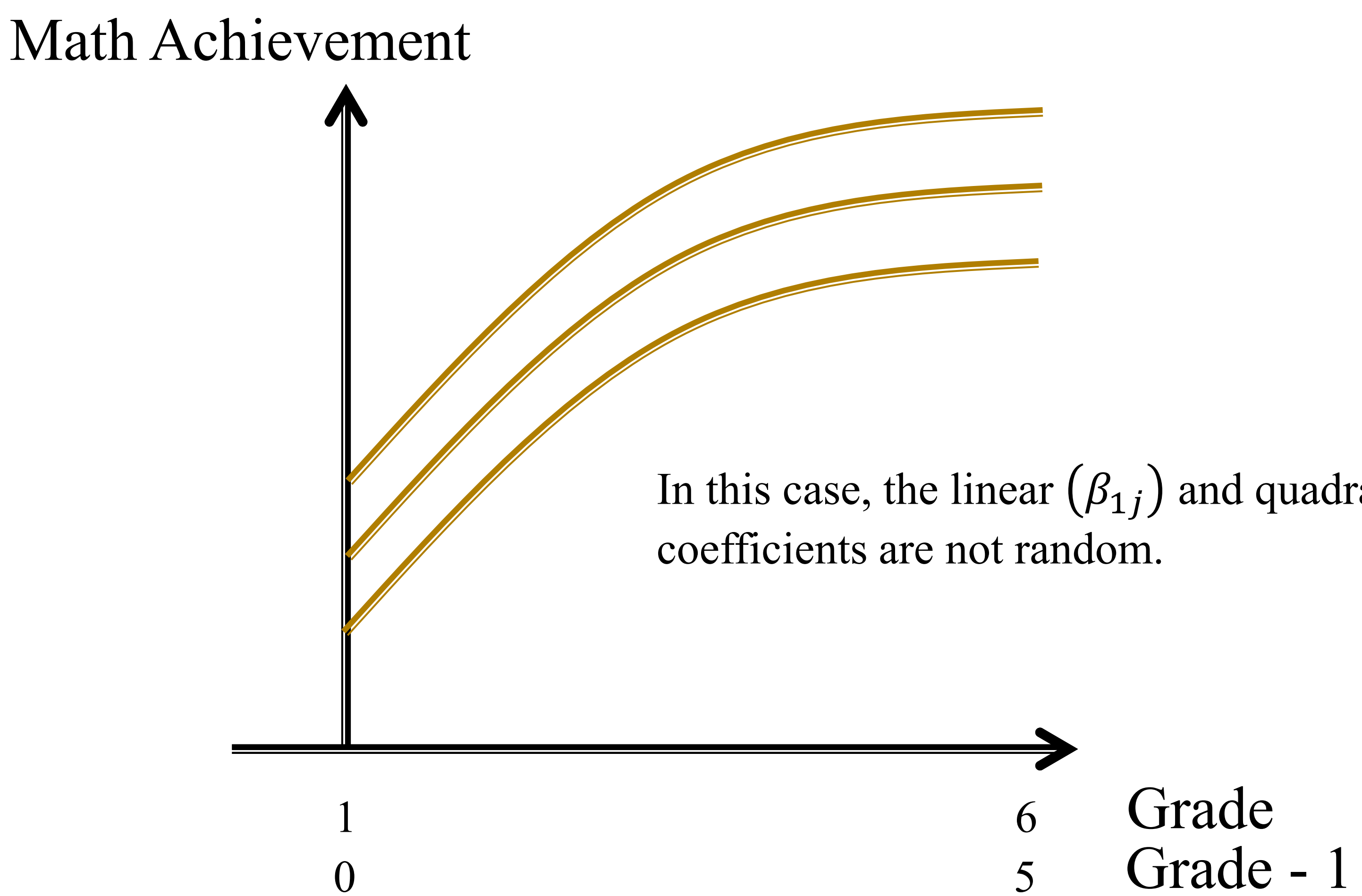
- What do all parameters represent?

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}(t_{ij} - 1)^2 + e_{ij}$$

L2: $$\beta_{0j} = 52 + u_{0j}$$
$$\beta_{1j} = 2 + u_{1j}$$
$$\beta_{2j} = -0.2 + u_{2j}$$

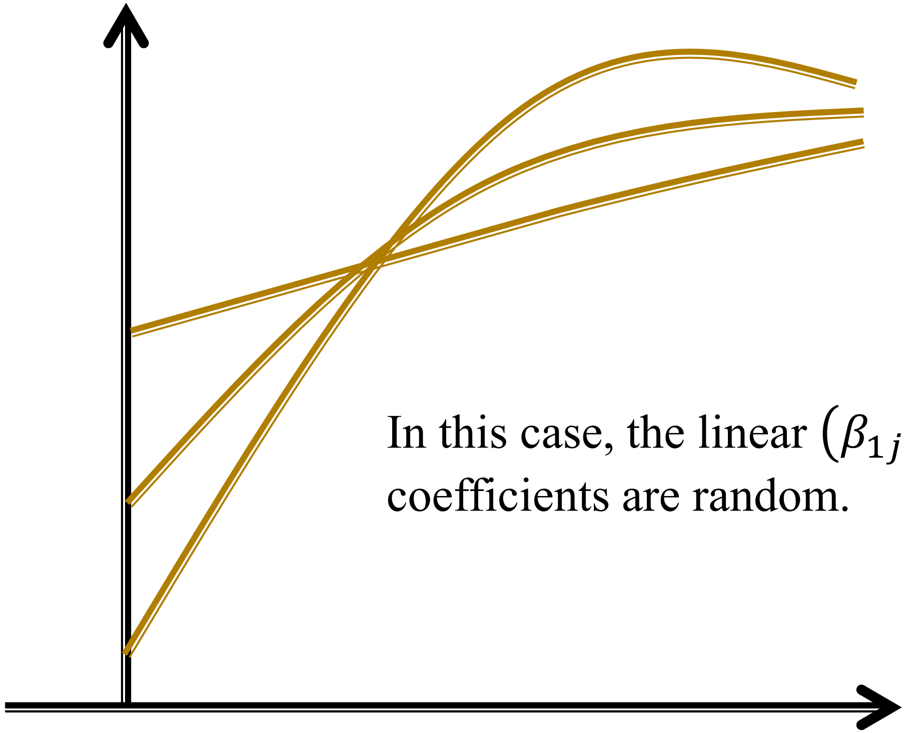
- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6)

Nonlinear Trajectory



In this case, the linear (β_{1j}) and quadratic (β_{2j}) coefficients are not random.

Nonlinear Trajectory



In this case, the linear (β_{1j}) and quadratic (β_{2j}) coefficients are random.

Nonlinear Trajectory

- Theoretical Justification
- Complex changes require data with more time points
- Three criteria for selecting a function
  - Model fit
  - Interpretability of the parameters
  - Behavior of the function matches expected changes in theory
- See Cudeck & Harring (2007)

Model with Covariates

- Time-varying covariate (Measurement-level predictors)
- Time-invariant covariate (Case-level predictors)
- The interpretations are similar to regular MLM.
- If the rate of change is moderated by time-invariant covariate, probing interaction can help.

Model with Covariates

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 2000) + e_{ij}$

L2: $\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$

- Y_{ij} = National GDP at Measurement *i* for Country *j* (in $Billion); t_{ij} = Year (2000-2012); W_j = European Countries (1 = Yes, 0 = No)

Model with Covariates

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 2000) + e_{ij}$

L2: $\beta_{0j} = 200 + 100W_j + u_{0j}$

$\beta_{1j} = 30 - 15W_j + u_{1j}$

- Y_{ij} = National GDP at Measurement *i* for Country *j* (in $Billion); t_{ij} = Year (2000-2012); W_j = European Countries (1 = Yes, 0 = No)

Model with Covariates

- What do all parameters represent?

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}(X_j - \bar{X}_{..}) + e_{ij}$$

L2: $$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$
$$\beta_{2j} = \gamma_{20} + u_{2j}$$

- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6); X_{ij} = Perceived Teacher Quality

Model with Covariates

- What do all parameters represent?

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}(X_j - \bar{X}_{..}) + e_{ij}$$

L2: $$\beta_{0j} = 52 + u_{0j}$$
$$\beta_{1j} = 2 + u_{1j}$$
$$\beta_{2j} = 1 + u_{2j}$$

- Y_{ij} = The math achievement score at Measurement *i* in Student *j*; t_{ij} = Grade (1-6); X_{ij} = Perceived Teacher Quality

Model with Covariates

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 12) + \beta_{2j}(X_j - \bar{X}_{..}) + e_{ij}$

L2: $\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$

$\beta_{2j} = \gamma_{20} + \gamma_{21}W_j + u_{2j}$

- Y_{ij} = Positive affect at Measurement *i* for Participant *j*; t_{ij} = Hour of a day (8:00-23:00); X_j = Stress level; W_j = Gender (1 = Female)

Model with Covariates

- What do all parameters represent?

L1: $$Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 12) + \beta_{2j}(X_j - \bar{X}_{..}) + e_{ij}$$

L2: $$\beta_{0j} = 40 - 10W_j + u_{0j}$$
$$\beta_{1j} = 0.01 + 1W_j + u_{1j}$$
$$\beta_{2j} = 2 + 2W_j + u_{2j}$$

- Y_{ij} = Positive affect at Measurement i for Participant j; t_{ij} = Hour of a day (8:00-23:00); X_j = Stress level; W_j = Gender (1 = Female)

Model with Covariates

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}X_j + e_{ij}$

L2: $\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$

$\beta_{2j} = \gamma_{20} + \gamma_{21}W_j + u_{2j}$

- Y_{ij} = Reaction Time at Measurement *i* for Participant *j*; t_{ij} = Trials (1-60); X_j = Language of Words (1 = Spanish; 0 = English); W_j = Country of Origin (1 = Spanish-Speaking Country; 0 = English)

Model with Covariates

- What do all parameters represent?

L1: $Y_{ij} = \beta_{0j} + \beta_{1j}(t_{ij} - 1) + \beta_{2j}X_j + e_{ij}$

L2: $\beta_{0j} = 100 + 200W_j + u_{0j}$

$\beta_{1j} = -0.5 + 0.001W_j + u_{1j}$

$\beta_{2j} = 200 - 420W_j + u_{2j}$

- Y_{ij} = Reaction Time at Measurement *i* for Participant *j*; t_{ij} = Trials (1-60); X_j = Language of Words (1 = Spanish; 0 = English); W_j = Country of Origin (1 = Spanish-Speaking Country; 0 = English)

References

- Cudeck, R., & Harring, J. R. (2007). Analysis of nonlinear patterns of changes with random coefficient models. *Annual Review of Psychology, 58*, 615-637.