Parameter Estimation and One-Sample t-Test

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Estimator

An estimator ($\hat{\theta}$) is a rule that tells you how to calculate an estimate of a population parameter using sample information.

Point Estimate

Interval Estimate

Two Properties of Good Estimators

1) Unbiased Estimator: $E(\hat{\theta}) = \theta$
2) Minimum Variance Estimator: least $Var(\hat{\theta})$

For example

The estimator of population mean (μ)

The estimator of population standard deviation (σ)

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{N}}$$

Degree of freedom (ν)

Sum of squared deviation

Author Note

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Comparing One-Sample Mean Unknown Population Standard Deviation

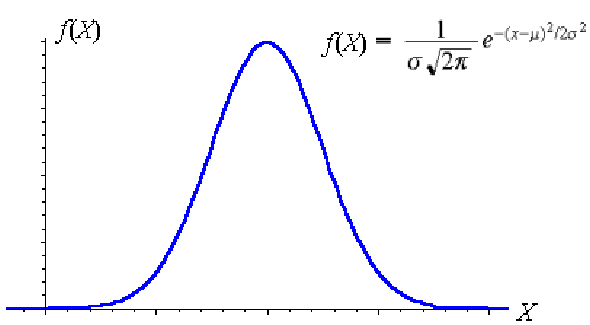
Population standard deviation is usually unknown.

Therefore, one sample z-test is not able to calculate.

One sample t-test can be used to evaluate hypotheses about a population mean

One sample z-test

1) Null hypothesis to be tested is
$$H_0 : \mu = \mu_0$$
2) Known population standard deviation
3) The z statistic is
$$z = \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} = \frac{X - \mu}{\frac{\sigma}{\sqrt{n}}}$$
$$z = \frac{\text{random variable} - \text{constant}}{\text{constant}}$$
4) The population is not necessary to be normal distribution
5) The sampling distribution is standard normal distribution (z distribution)



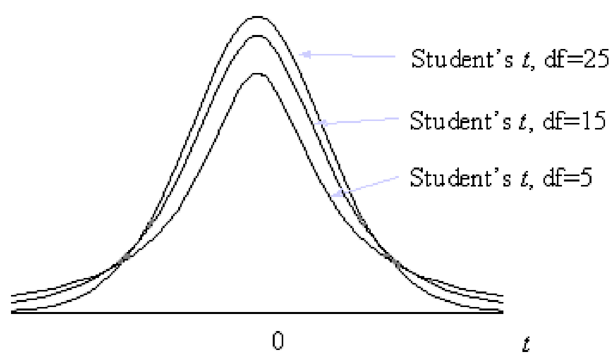
6) Find critical value, p value in areas under standard normal distribution (Table D2, Kirk, 2008) or function in MS Excel

= NORMSDIST(z)

If $z < 0$, one-tailed p value =
NORMSDIST(z)

One sample t-test

1) Null hypothesis to be tested is
$$H_0 : \mu = \mu_0$$
2) Unknown population standard deviation, instead, use standard deviation estimator (random variable)
3) The t statistic is
$$t = \frac{\bar{X} - \mu_{\bar{X}}}{\hat{\sigma}_{\bar{X}}} = \frac{X - \mu}{\frac{\hat{\sigma}}{\sqrt{n}}}$$
$$t = \frac{\text{random variable} - \text{constant}}{\text{random variable}}$$
4) The population should be normal distribution, but t statistic robust form nonnormality
5) The sampling distribution is Student's t distribution with $df = n - 1$



The t and z distributions are so similar for samples equal to or larger than 30.

6) Find critical value, p value in areas under Student's t distribution (Table D3, Kirk, 2008) or function in MS Excel

= TDIST(t, df, tails)

If $z > 0$, one-tailed p value = 1 – NORMDIST(z)
Two-tailed p value = two times of one tailed p value

Hypothesis testing on sampling distribution of the mean by t statistic

Example 1 Are 4 people with average height equal to 160 cm ($\hat{\sigma}$ = 8 cm) Thai men, if they are same sex and same nationality?

On average, Thai men are tall 170 cm.

Research Hypothesis: These four people are not Thai men

Alternative Hypothesis: These four people are not Thai men or $H_1: \mu < 170$

Null Hypothesis: These four people are Thai men or $H_0: \mu \geq 170$

Alpha = .05

Building the characteristic of t distribution of Thai men sample means with $df = n - 1 = 3$

The critical value when alpha = .05 (one-tailed), from Table D3, is -2.353.

This distribution has $\mu_{\bar{X}} = 170$ and $\hat{\sigma}_{\bar{X}} = 8/\sqrt{4} = 4$

Transform the sample mean to t statistic

$$t = \frac{\bar{X} - \mu_{\bar{X}}}{\hat{\sigma}_{\bar{X}}} = \frac{160 - 170}{4} = -2.5$$

The t statistic falls in critical region. Therefore, reject null hypothesis.

These four people are not Thai men.

Example 2 Last 3 years, the average IQ of Thai children was 100. In this year, the psychologist notices that the average IQ changes. He randomly collects 100 Thai children IQ. The average IQ is 98 and standard deviation is 12. Does Thai children IQ really change or by chance?

Research Hypothesis: Thai children IQ really changes.

Alternative Hypothesis: Thai children IQ changes or $H_1: \mu \neq 100$

Null Hypothesis: Thai children IQ does not change or $H_0: \mu = 100$

Alpha = .05

Building the characteristic of *t* distribution of Thai men sample means with *df* = *n* - 1 = 99

This distribution has $\mu_{\bar{X}} = 100$ and $\hat{\sigma}_{\bar{X}} = 12/\sqrt{100} = 1.2$

Transform the sample mean to *t* statistic

$$t = \frac{\bar{X} - \mu_{\bar{X}}}{\hat{\sigma}_{\bar{X}}} = \frac{98 - 100}{1.2} = -1.67$$

Find out the *p* value of this *t* value by MS Excel

= TDIST(1.67,99,2) = .0981

The probability of IQ mean less than or equal to 98, when n = 100 and null hypothesis is true, is 9.81 %, not less than alpha.

The psychologist cannot conclude that Thai children IQ really change.

This process is called one sample *t*-test.

Assumption of One Sample *t* Test

In using t statistics and t sampling distribution, it assumes that

1) A random sample of n observations is obtained from the population of interest.
2) The population is normally distributed
3) The population standard deviation is unknown

Relationship between sample size and hypothesis testing with sample mean

From sampling distribution formula

$$t = \frac{\bar{X} - \mu_{\bar{X}}}{\hat{\sigma}_{\bar{X}}} = \frac{X - \mu}{\frac{\hat{\sigma}}{\sqrt{n}}}$$

If *n* increase, *t* will increase. Then, the chance to reject null hypothesis will increase.

The power of rejection false H_0 depends on

1) sample size
2) effect size or difference between $\bar{X}$ and μ
3) sample standard deviation
4) significance level
5) type of test: one-tailed or two-tailed

General Form of *z* and *t* Statistics

The general formula of z statistics

$$z = \frac{\text{Statistics} - \text{Null hypothetical value}}{\text{Standard Error of Statistics}}$$

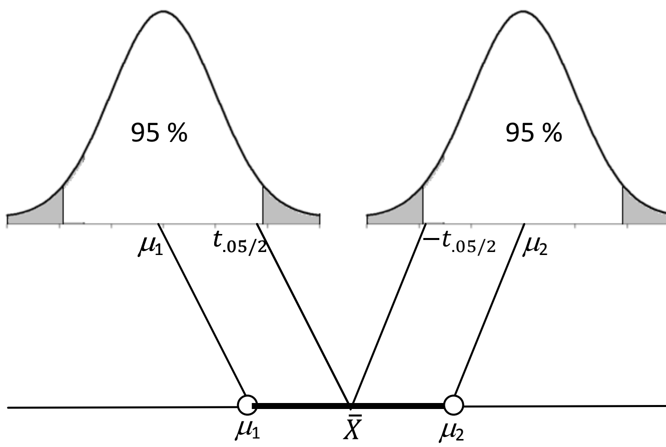
The general formula of t statistics

$$t = \frac{\text{Statistics} - \text{Null hypothetical value}}{\text{Estimated Standard Error of Statistics}}$$

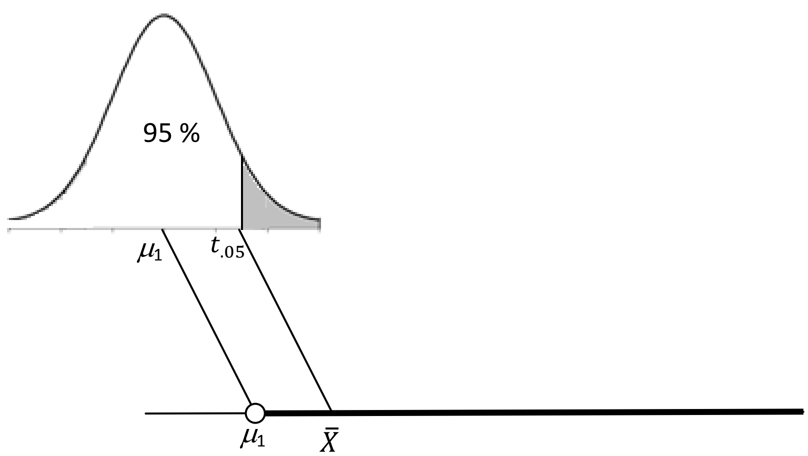
Confidence Interval Say about inferential stat again

The point estimate of particular sample is unlikely to equal the population parameter.

Interval estimate, called confidence interval, is the segment that the population mean has a high probability of lying on.



Two-tailed CI 95 %



One-tailed CI 95 %

Upper and lower endpoints

Confidence Interval can either one-sided or two-sided interval.

Confidence Coefficient (specified by researcher)

Interval Estimate from sample mean by *t* distribution

What is the interval estimate from sample mean and standard deviation to population mean?

$$\text{Prob}(-t_{.05/2,\nu} < t < t_{.05/2,\nu}) = 1 - .05 = .95$$

$$\text{Prob}\left(-t_{.05/2,\nu} < \frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} < t_{.05/2,\nu}\right) = .95$$

$$\text{Prob}\left(\frac{-t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}} < \bar{X} - \mu < \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

$$\text{Prob}\left(-\bar{X} - \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}} < -\mu < -\bar{X} + \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

$$\text{Prob}\left(\bar{X} + \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}} > \mu > \bar{X} - \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

The general form of a two side 100(1 - α) % confidence interval for μ is

$$\text{Prob}\left(\bar{X} - \frac{t_{\alpha/2,\nu}\hat{\sigma}}{\sqrt{n}} < \mu < \bar{X} + \frac{t_{\alpha/2,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

The general form of a one side 100(1 - α) % confidence interval for μ is

$$\text{Prob}\left(\bar{X} - \frac{t_{\alpha,\nu}\hat{\sigma}}{\sqrt{n}} < \mu\right) = .95 \quad \text{or} \quad \text{Prob}\left(\mu < \bar{X} + \frac{t_{\alpha,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

Example of Interval Estimate from sample mean by *t* distribution

<u>Example 1</u> The psychologist randomly collects 100 Thai children IQ. The average IQ is 98 and standard deviation is 12. What is the two-sided 95 % confidence interval for their population mean?

From the sample, $\bar{X} = 98$, $\hat{\sigma} = 12$, $n = 100$.

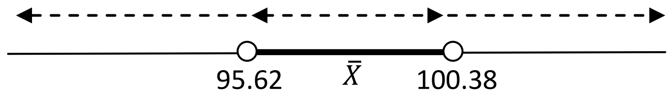
$t_{.05/2,99} = 1.984$

$$\text{Prob}\left(\bar{X} - \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}} < \mu < \bar{X} + \frac{t_{.05/2,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

$$\text{Prob}\left(98 - \frac{1.984 \times 12}{\sqrt{100}} < \mu < 98 + \frac{1.984 \times 12}{\sqrt{100}}\right) = .95$$

$$\text{Prob}(95.6192 < \mu < 100.3808) = .95$$

Reject Fail to Reject Reject



Two-tailed CI 95 %

If null hypothesis to be tested is $H_0: \mu = 100$, fail to reject null hypothesis.

<u>Example 2</u> Four people with average height equal to 160 cm ($\hat{\sigma}$ = 8 cm). What is the one-sided 95 % confidence interval for population mean, if research hypothesis is that their population mean is less than specific value?

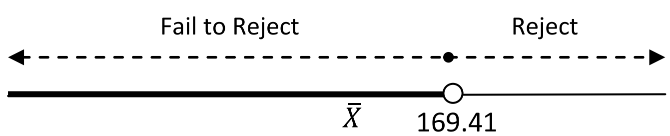
From the sample, $\bar{X} = 160$, $\hat{\sigma} = 8, n = 4.$

$t_{.05,3} = 2.353$

$$\text{Prob}\left(\mu < \bar{X} + \frac{t_{.05,\nu}\hat{\sigma}}{\sqrt{n}}\right) = .95$$

$$\text{Prob}\left(\mu < \underset{\bar{X}}{160} + \frac{2.353 \times 8}{\sqrt{4}}\right) = .95$$

$$\text{Prob}(\mu < 169.412) = .95$$



One-tailed CI 95 %

If null hypothesis to be tested is $H_0: \mu \geq 170$, reject null hypothesis.

Assumption of Interval Estimate from sample mean by *t* distribution

In using t statistics and t sampling distribution, it assumes that

1) A random sample of *n* observations is obtained from the population of interest.
2) The population is normally distributed
3) The population standard deviation is unknown

Interval Estimate from sample proportion

What is the interval estimate from sample proportion ($\hat{p}$) to population proportion (p)?

Estimated standard error of proportion

$$\hat{\sigma}_p = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

It is different from standard error in binomial distribution.

$$\sigma_p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{p(1-p)}{n}}$$

This standard error uses the proportion in null hypothesis. However, confidence interval uses sample statistic to estimate parameter, so it uses $\hat{p}$.

The general form of a two side 100(1 - α) % confidence interval for μ is

$$\text{Prob}\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = .95$$

The general form of a one side 100(1 - α) % confidence interval for μ is

$$\text{Prob}\left(\hat{p} - z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p\right) = .95 \quad \text{or} \quad \text{Prob}\left(p < \hat{p} + z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = .95$$

Example of Interval Estimate from sample proportion

Example 1 Flipping the coin 400 times found 50 heads. What is the two-sided 95 % confidence interval for their population proportion?

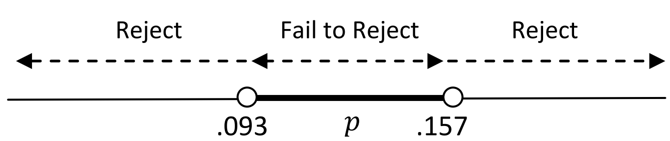
From the sample, $\hat{p} = 0.125$, n = 400.

$z_{.05/2}$ = 1.96

$$\text{Prob}\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = .95$$

$$\text{Prob}\left(0.125 - 1.96\sqrt{\frac{0.125(1-0.125)}{400}} < p < 0.125 + 1.96\sqrt{\frac{0.125(1-0.125)}{400}}\right) = .95$$

$$\text{Prob}(0.093 < p < 0.157) = .95$$



Two-tailed CI 95 %

If null hypothesis to be tested is $H_0: p = 0.5$, reject null hypothesis.

In conclusion, the coin is bias.

Example 2 Last year, the market share of detergent A is 70 %. In this year, the company use marketing strategy to boost up the market share. From survey research that sample size = 200, 160 respondents choose detergent A. What is the one-sided 95 % confidence interval for population proportion?

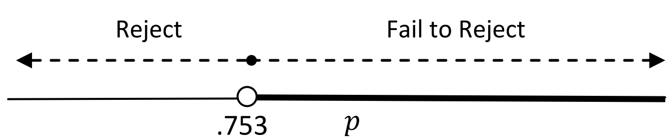
From the sample, $\hat{p} = 0.80$, n = 200.

$z_{.05}$ = 1.645

$$\text{Prob}\left(\hat{p} - z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p\right) = .95$$

$$\text{Prob}\left(p > 0.8 - 1.645\sqrt{\frac{0.8(1-0.8)}{200}}\right) = .95$$

$$\text{Prob}(p > 0.753) = .95$$



One-tailed CI 95 %

If null hypothesis to be tested is $H_0: p \leq .7$, reject null hypothesis.

Assumption of Interval Estimate from sample proportion

In approximating area by normal distribution in binomial test, it assumes that

1) Random sampling from the population of interest
2) Binomial Population
3) np and $n(1- p)$ are both greater than 15
4) The population is at least 10 times larger than the sample.

Reading Confidence Interval

Yes. A 95% confidence interval for p is from .093 to .157.

Yes. The degree of your confidence that p lies in the open interval from .093 to .157 is .95.

No. The probability of p lies in lies in the open interval from .093 to .157 is .95.

Instead, the probability is either 0 or 1.

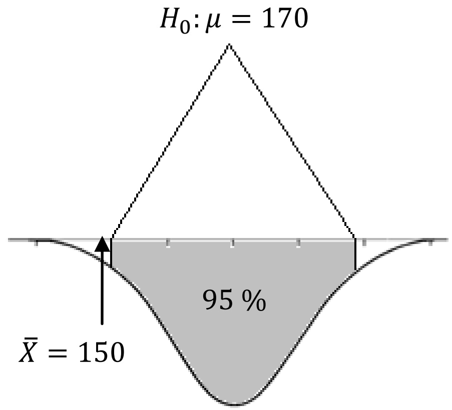
The probability that a randomly selected interval from this infinite set will contain p is .95.

Inferential Statistics

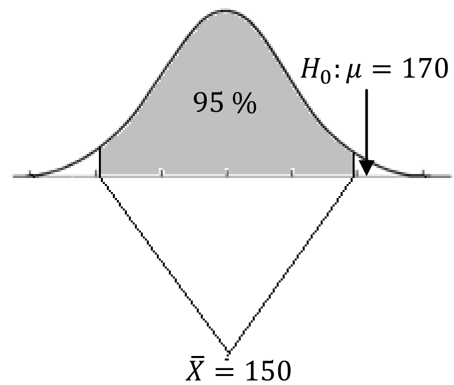
There are two ways for inferring sample statistics to population parameter.

1) Hypothesis Testing (such as in binomial distribution, sampling distribution of the mean)
2) Parameter Estimation (such as confidence interval of population proportion, population mean)

Hypothesis Testing



Confidence Interval



Criticism on Null Hypothesis Significance Testing

1) A null hypothesis significance test addresses the question “Is chance a likely explanation for the results that have been obtained?” The test does not address the question “Are the results important or useful?”
2) In scientific inference, you want to know $\text{Prob}(H_0|D)$ not $\text{Prob}(D|H_0)$. See example in Cohen, J. (1994). The earth is round (p < .05). *American Psychologist, 49*, 997-1003.
3) A null hypothesis significance testing is a trivial exercise. All null hypotheses are false, especially in two-tailed test.
4) Significance testing adopt a fixed significance level such as $\alpha = .05$, a researcher turns a continuum of uncertainty into a dichotomous reject-do-not-reject decision.

The advantage of confidence interval

1) The confidence interval provides the information about parameter estimation from statistic.
2) It can be used to test any null hypothesis by looking at the interval.

Why hypothesis is still popular?

1) In the past, hypothesis significance testing has been the dominant approach to statistical inference.
2) Reject or fail to reject null hypothesis decision likes reject or accept researcher hypothesis.
3) Some statistical inference cannot be addressed using confidence interval.

Effect Size and Practical Significance

Statistical significance tells you whether null hypothesis is tenable. However, it does not tell you whether the difference is large enough for practicality.

Effect magnitude statistics can assist a researcher deciding whether results are practically significant.

Most effect magnitude statistics fall into one of two categories:

1) Measures of effect size
2) Measures of strength of association

Cohen's *d* measures of effect size

$$d = |\mu - \mu_0|/\sigma$$

d = 0.2 is a small effect.

d = 0.5 is a medium effect.

d = 0.8 is a large effect.

Hedges' estimate of effect size for one-sample *z* or *t* test

$$g = \frac{|\bar{X} - \mu_0|}{\hat{\sigma}}$$

$$g = t/\sqrt{n}$$

Identifying Sample Sizes

Sometimes, researchers want to know sample size.

There are two criteria for determining sample size.

1) Parameter estimation
2) Statistical Significance in desired effect size.

Parameter Estimation

If sample size increases, the margin of error in confidence interval expands.

However, the number of sample size depends on random sampling techniques.

In sample size estimation, you must consider

1) Precision (error in estimation)

$$|\hat{\theta} - \theta| < e$$

2) Reliability or Confident Coefficient

$$P\left(\left|\hat{\theta} - \theta\right| < e\right) = 1 - \alpha$$

Simple Random Sampling

Find sample sizes for desired precision and desired reliability

$$P\left(\frac{\left|\hat{\theta} - \theta\right|}{\sigma_{\hat{\theta}}} > \frac{e}{\sigma_{\hat{\theta}}}\right) = \alpha$$

$$P\left(|Z| > \frac{e}{\sigma_{\hat{\theta}}}\right) = \alpha = P\left(|Z| > Z_{\alpha/2}\right)$$

$$\frac{e}{\sigma_{\hat{\theta}}} = Z_{\alpha/2}$$

$$\frac{e^2}{Var(\hat{\theta})} = Z_{\alpha/2}^2$$

$$e^2 = Z_{\alpha/2}^2 Var(\hat{\theta})$$

Find sample size for estimating <u>population mean</u>

$$Var\left(\hat{\theta}\right) = \frac{\hat{\sigma}^2}{n}$$

$$e^2 = Z_{\alpha/2}^2 \frac{\hat{\sigma}^2}{n}$$

$$n_0 = \frac{Z_{\alpha/2}^2 \hat{\sigma}^2}{e^2}$$

Find sample size for estimating <u>population proportion</u>

$$Var\left(\hat{\theta}\right) = \frac{p(1-p)}{n}$$

$$e^2 = Z_{\alpha/2}^2 \frac{p(1-p)}{n}$$

$$n_0 = \frac{Z_{\frac{\alpha}{2}}^2 p(1-p)}{e^2}$$

Finite population correction

$$n = \frac{n_0}{1 + \frac{n_0}{N}}$$

Remarks

1) If determine e = 0, n = N.
2) If you don't know some statistics, you may tryout survey or draw from previous research.
3) The tryout sample size should be more than 30.

Statistical Significance for Desired Effect Size

If sample size increases, the probability of reject null hypothesis increases.

In sample size estimation, you must consider

1) Desired level of effect size (*d*) or practical difference
2) Desired probability of type one error (α)
3) Desired probability of type two error (β)
4) One-tailed or two-tailed test

In one sample *t*-test, the sample size required for statistical significance is shown in Appendix D8 (Kirk, 2008) or calculated from

$$n = \frac{(z_\alpha - z_\beta)^2 \hat{\sigma}^2}{(\mu - \mu_0)^2}$$

$$n = \frac{(z_\alpha - z_\beta)^2}{d^2}$$

In binomial test, the sample size required for statistical significance is calculated from

$$n = \frac{(z_\alpha \sqrt{p_0 q_0} - z_\beta \sqrt{pq})^2}{(p - p_0)^2}$$

For example $p - p_0 = 0.1;\ \alpha = .05; \beta = .20$

p_0	p	z_α	z_β	Required n
0.4	0.5	-1.96	0.84	194
0.3	0.4	-1.96	0.84	181
0.2	0.3	-1.96	0.84	152
0.1	0.2	-1.96	0.84	107

The more extreme, the less sample size required.